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On the Dimension of Bivariate Spline Spaces on Generalized Quasi-cross-cut Partitions

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We consider spaces of piecewise polynomials of degree n and smoothness $k<n$, defined over a rectilinear partition of a simply connected domain of $\mathbb{R}^2$. We prove that the dimension of the space agrees with Schumaker's lower bound if $n \geqslant k+2\lceil (k+1)/(\eta-1)\rceil-2$ and $\eta \geqslant 2$, where η depends on the structure of the partition.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and $\Delta = \{\Omega_i, i=1, ..., \omega\}$ a partition of Ω. Here and throughout, we shall assume Δ a rectilinear partition of Ω, i.e., for each i, $\partial\Omega_i$ is homeomorphic to a circle and $\partial\Omega_i \cap \mathring{\Omega}$ is a piecewise linear curve.

We are interested in the space of bivariate splines of degree n and smoothness k, $n>k\geqslant 0$, associated with the given partition

$$S_n^k(\Omega, \Delta) = \{s : s \in C^k(\Omega), s_{|\Omega_i} \in \mathbb{P}_n, \forall \Omega_i \in \Delta\},$$

where $\mathbb{P}_n$ is the $(n+1)(n+2)/2$ dimensional linear space of polynomials of total degree n.

In recent years there has been considerable work on identifying the dimension of the spline spaces $S_n^k(\Omega, \Delta)$ ([13, 14, 6, 2] and references therein).

For general values of n and k for arbitrary partitions both lower and upper bounds on the dimension are known [10, 14]. If Δ is a triangulation, dimension formulae have been established in the cases $n \geqslant 3k+2$ [8–10], $n=4$, $k=1$ [1], $n=3k+1$ for non-degenerate triangulations [2].

With regard to partitions which are not necessarily triangulations, formulae for the dimensions have been given for quasi-cross-cut partitions [5] and for general rectilinear partitions if $n \leqslant k+(k+1)/D$, where $D+1$ is the

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maximum number of edges with different slopes emanating from an interior vertex Δ [12].

The spaces $S_n^k(\Omega, \Delta)$ with n "large enough with respect to k and Δ" are the most interesting both for their approximation properties and for containing non-trivial elements with compact support [6]. Even for these spaces several results are known when Δ is a triangulation [9, 3], while there are still several unsolved problems if Δ is a general rectilinear partition. In this case also the dimension of the space presents a more subtle geometric dependence [7].

In this paper we investigate the dimension problem for the spline space defined over general rectilinear partitions. In particular we present new bounds for the dimension and we prove that its value agrees with the lower bound given in [14] if Δ is a generalization of a quasi-cross-cut partition [5] and n is "large enough" with respect to k and to the number of cross-cuts and rays traversing the interior vertices of Δ.

To establish these results we shall consider the usual cartesian coordinates, in fact the barycentric coordinates and Bezier–Bernstein form for multivariate polynomials are not useful if the cells of Δ are not necessarily triangles.

2. Main Results

We introduce some notation. Given the partition Δ the straightline segments making up $\partial\Omega_i \cap \partial\Omega_j$, $i \neq j$, $i, j = 1, ..., \omega$, shall be called edges, and the points where the edges join each other or meet $\partial\Omega$ shall be called vertices (note that, from the definition, here we consider interior edges only). Let $P_i = (x_i, y_i)$, $i = 1, ..., V$, be the vertics of Δ and P_i, $i = 1, ..., v < V$, the interior vertices. For $i = 1, ..., v$,

I_i $= \{j : P_j$ is adjacent to P_i, $1 \leqslant j \leqslant V\}$,

l_{is} = oriented edge joining P_i to P_s,

ε_i = number of edges of Δ emanating from P_i,

e_i = number of edges of Δ emanating from P_i with different slopes,

E = number of edges of Δ,

N_c^i = number of cross-cuts (i.e., line segments with both endpoints on $\partial\Omega$) crossing P_i,

F^i = number of rays (i.e., line segments joining an interior vertex to $\partial\Omega$) crossing P_i,

η_i $= N_c^i + F^i$,

$\eta \quad = \min\{\eta_i, i = 1, ..., v\}$,

$J_i \quad = \begin{cases} \lceil (k+1)/(\eta_i - 1) \rceil & \text{if } \eta_i \geqslant 2, \\ +\infty & \text{otherwise,} \end{cases}$

E_d = number of edges joining two interior vertices,

E_{cd} = number of edges joining two interior vertices, overlying a cross-cut or a ray,

$I^d \quad = \{(i, s) \in \mathbb{N}^2, \max(\eta_i, \eta_s) \geqslant 2, 1 \leqslant i < s \leqslant v : l_{is}$ does not overlie a cross-cut or a ray$\}$,

$\beta \quad = (n-k)(n-k+1)\frac{1}{2}$,

$\alpha \quad = (n+1)(n+2)\frac{1}{2}$,

$\phi \quad = (k+1)(k+2)\frac{1}{2}$,

where $\lceil x \rceil$ denotes the smallest integer greater or equal to x.

Definition 2.1. Δ is called a generalized quasi-cross-cut partition provided that $\eta \geqslant 2$.

We shall prove the following results:

Theorem 2.1. *Let Δ be a partition of a simply connected domain $\Omega \subset \mathbb{R}^2$, then*

$$\dim S_n^k(\Omega, \Delta) \leqslant \alpha + \beta(E + E_d - E_{cd}) - \gamma - \sum_{(i,s) \in I^d} n_{is}, \tag{2.1}$$

where

$$\gamma = \sum_{j=1}^{n-k} \sum_{i=1}^{v} \min(k+1+j, je_i) = v(\alpha - \phi) - \sum_{j=1}^{n-k} \sum_{i=1}^{v} (k+1+j-je_i)_+,$$

$$n_{is} = \begin{cases} \beta - \frac{1}{2}(J_i + J_s - n + k - 1)(J_i + J_s - n + k - 2)_+, \\ \qquad \text{if} \quad J_i, J_s \leqslant n-k, \\ \beta - \frac{1}{2}(\theta - 1)\theta, \\ \qquad \text{if} \quad \theta = \min(J_i, J_s) \leqslant n-k, \max(J_i, J_s) > n-k, \\ 0, \qquad \text{if} \quad J_i, J_s > n-k, \end{cases}$$

$$(x)_+ = \max(0, x).$$

Theorem 2.2. *Let Δ be a generalized quasi-cross-cut partition of a simply connected domain $\Omega \subset \mathbb{R}^2$, if*

$$n \geqslant k - 2 + 2\left\lceil \frac{k+1}{\eta - 1} \right\rceil, \tag{2.2}$$

then

$$\dim S_n^k(\Omega, \Delta) = \alpha + \beta E - \gamma.$$

3. Conformality Conditions

It is well known [4] that an element of $S_n^k(\Omega, \Delta)$ is determined by one polynomial of total degree n and by $2E$ polynomials, $q_{ij} \in \mathbb{P}_{n-k-1}$, which must satisfy the following conformality conditions [5]:

$$\sum_{j \in I_i} [l_{ij}(x, y)]^{k+1} q_{ij}(x, y) \equiv 0, \qquad i = 1, ..., v, \tag{3.1}$$

$$q_{ij} \equiv -q_{ji}, \tag{3.2}$$

where

$$l_{ij}(x, y) = l_{ji}(x, y) = a_{ij}x + b_{ij}y - (a_{ij}x_i + b_{ij}y_i) = 0,$$
$$(a_{ij})^2 + (b_{ij})^2 > 0,$$

is the equation of the straight line containing the edge l_{ij}.

Conditions (3.1), (3.2) determine the dimension of $S_n^k(\Omega, \Delta)$. In order to rewrite them in a more convenient form we consider the translation

$$\begin{cases} \xi = x - x_i \\ \sigma = y - y_i \end{cases} \tag{3.3}$$

and the differential operators

$$D_{0,i} = I,$$
$$D_{1,i} = x_i \frac{\partial}{\partial \xi} + y_i \frac{\partial}{\partial \sigma},$$
$$D_{j,i} = D_{1,i} D_{j-1,i}, \qquad j = 2, ...,$$

where I denotes the identity operator.

From the Taylor expansion it immediately follows:

Lemma 3.1. *Given the translation* (3.3), *if* $q(x, y) \in \mathbb{P}_d$ *and*

$$p(\xi, \sigma) = q(x(\xi), y(\sigma)),$$

then

$$p(\xi, \sigma) = \sum_{j=0}^{d} \frac{1}{j!} D_{j,i} q(\xi, \sigma).$$

Let us consider the linear operator $L_i : \mathbb{P}_{n-k-1} \to \mathbb{P}_{n-k-1}$,

$$L_i = I + D_{1,i} + \frac{1}{2!} D_{2,i} + \cdots + \frac{1}{(n-k-1)!} D_{n-k-1,i}.$$

L_i is an isomorphism in $\mathbb{P}_{n-k-1}$.

Considering at each interior vertex of Δ a translation as (3.3) and using for simplicity the same symbols for the independent variables, condition (3.1) becomes

$$\sum_{j \in I_i} [a_{ij}x + b_{ij}y]^{k+1} L_i q_{ij}(x, y) \equiv 0, \quad i = 1, ..., v,$$

while (3.2) is unchanged. Let

$$L_i q_{ij}(x, y) = p_{ij}(x, y) = \sum_{r=0}^{n-k-1} p_{ij}^{(r)}(x, y),$$

where $p_{ij}^{(r)}$ is the homogeneous component of p_{ij} of degree r.

Denoting by L_i^{-1} the inverse operator of L_i in $\mathbb{P}_{n-k-1}$, from (3.2)

$$L_i^{-1} p_{ij}(x, y) \equiv -L_j^{-1} p_{ji}(x, y),$$

and

$$L_i L_j L_i^{-1} p_{ij}(x, y) \equiv -L_i L_j L_j^{-1} p_{ji}(x, y),$$

finally, observing that $L_i L_j = L_j L_i$ (in fact the derivatives commute in $\mathbb{P}_{n-k-1}$) we have

$$L_j p_{ij}(x, y) + L_i p_{ji}(x, y) \equiv 0, \tag{3.4}$$

$$\sum_{j \in I_i} [a_{ij}x + b_{ij}y]^{k+1} p_{ij}(x, y) \equiv 0, \qquad i = 1, ..., v. \tag{3.5}$$

4. The Linear System

System (3.4)–(3.5) involves $2E\beta$ unknowns, but $\beta(E - E_d)$ of them, associated with the edges emanating from boundary vertices, appear in (3.4) only, so they are determined by the others explicitly. Then

$$\dim S_n^k(\Omega, \Delta) = \alpha + \beta(E + E_d) - \operatorname{rank} \mathscr{M}, \tag{4.1}$$

where $\mathcal{M}$ is the matrix of the linear system

$$\sum_{j \in I_i} [a_{ij}x + b_{ij}y]^{k+1} p_{ij}(x, y) \equiv 0, \qquad i = 1, ..., v, \tag{4.2}$$

$$L_j p_{ij}(x, y) + L_i p_{ji}(x, y) \equiv 0, \qquad 1 \leqslant i < j \leqslant v. \tag{4.3}$$

With a suitable arrangement of the equations and unknowns, recalling the form of $p_{ij}(x, y)$, $\mathcal{M}$ has the structure

$$\mathcal{M} = \begin{bmatrix} \mathbf{M}^1 & 0 & \cdots & 0 \\ \vdots & \mathbf{M}^2 & & \vdots \\ 0 & & \cdots & \mathbf{M}^v \\ & \mathbf{L} & & \end{bmatrix}.$$

Each $\mathbf{M}^i$, $i = 1, ..., v$, is the diagonal block matrix of the equations (4.2) related to the vertex P_i and it captures the influence of the edges emanating from this vertex. More precisely

$$\mathbf{M}^i = \begin{bmatrix} \mathbf{M}^i_{n-k} & & \cdots & 0 \\ & \mathbf{M}^i_{n-k-1} & & \\ \vdots & & & \\ 0 & & \cdots & \mathbf{M}^i_1 \end{bmatrix},$$

where $\mathbf{M}^i_r$ is the $(r+k+1)$ by $r\varepsilon_i$ matrix containing the equations (4.2) involving $p^{(r-1)}_{ij}(x, y)$, $j \in I_i$. To each couple of collinear edges crossing P_i corresponds a couple of blocks of equal columns in $\mathbf{M}^i_r$ and [15]

$$\operatorname{rank} \mathbf{M}^i_r = \min(k + 1 + r, re_i).$$

In the following with the term *columns of* $\mathbf{M}^i_r$ *associated to the edge* l_{ij} we will refer to the columns of $\mathbf{M}^i_r$ corresponding to the coefficients of $p^{(r-1)}_{ij}(x, y)$.

The matrix $\mathbf{L}$ contains equations (4.3), so it controls the interior vertices interaction. In order to investigate its structure we consider the set

$$\{y^{n-k-1}, y^{n-k-2}x, ..., x^{n-k-1}, ..., y, x, 1\}, \tag{4.4}$$

as a basis of $\mathbb{P}_{n-k-1}$.

If $\mathbf{L}_i$ denotes the matrix of L_i with respect to this basis, then $\mathbf{L}_i$ is a lower triangular block matrix

$$\mathbf{L}_i = \begin{bmatrix} \mathbf{I} & & & \\ \mathbf{B}^{1,n-k-1}_i & \mathbf{I} & & \\ \vdots & & \ddots & \\ \mathbf{B}^{n-k-1,n-k-1}_i & \cdots & \mathbf{B}^{1,1}_i & \mathbf{I} \end{bmatrix},$$

where $\mathbf{B}_i^{r,s}$ is the $(s+1-r)$ by $(s+1)$ matrix representing the operator

$$\frac{1}{r!}D_{r,i}: \langle y^s, ..., x^s\rangle \to \langle y^{s-r}, ..., x^{s-r}\rangle.$$

Then

$$\mathbf{L} = \begin{bmatrix} \Lambda_1 \\ \vdots \\ \Lambda_{E_d} \end{bmatrix},$$

where each Λ_s is a matrix with β rows and it contains the equations (4.3) for two polynomials p_{ir}, p_{ri} associated with an edge joining two interior vertices. More precisely, the only non-zero columns in Λ_s are those corresponding to the columns of $\mathbf{M}^i$ ($\mathbf{M}^r$) related to $p_{ir}(p_{ri})$: the first $(n-k)$ columns of $\mathbf{L}_r$ are aligned with the $(n-k)$ columns of $\mathbf{M}^i_{n-k}$ associated to $p_{ir}^{(n-k-1)}(x, y)$ and so on.

The following lemma holds:

LEMMA 4.1. *Let t_i, $t_r \in \mathbb{N}$ be such that $1 \leqslant t_i + t_r \leqslant n-k = d+1$ and $\mathbf{B}_{ir} = (\mathbf{B}_i | \mathbf{B}_r)$ the submatrix of $(\mathbf{L}_i | \mathbf{L}_r)$, where*

$$\mathbf{B}_i = \begin{bmatrix} \mathbf{I} & \cdots & 0 \\ \mathbf{B}_i^{1,d} & & \vdots \\ \vdots & \ddots & 0 \\ \mathbf{B}_i^{t_i-1,d} & \cdots & \mathbf{I} \\ \vdots & & \vdots \\ \mathbf{B}_i^{t_i+t_r-1,d} & \cdots & \mathbf{B}_i^{t_r,d+1-t_i} \end{bmatrix},$$

and $\mathbf{B}_r$ is defined analogously by interchanging i with r. Then, if $(x_i, y_i) \neq (x_r, y_r)$, $\mathbf{B}_{ir}$ has maximum rank, i.e.,

$$\operatorname{rank} \mathbf{B}_{ir} = \sum_{j=0}^{t_i+t_r-1} (d+1-j).$$

Proof. Let us consider the space $\mathbb{P}_d \times \mathbb{P}_d$, the linear subspace of $\mathbb{P}_d$,

$$\mathbb{P}_d^s = \langle y^d, y^{d-1}x, ..., x^d, ..., y^s, ..., x^s\rangle,$$

and the linear operator,

$$B_{ir}: \mathbb{P}_d^{d+1-t_i} \times \mathbb{P}_d^{d+1-t_r} \to \mathbb{P}_d^{d+1-t_i-t_r};$$

$$B_{ir}(p, q) = \mathscr{P}^{d+1-t_i-t_r}\left(\sum_{j=0}^{t_i+t_r-1} \frac{1}{j!} D_{j,i}(p) + \frac{1}{j!} D_{j,r}(q)\right),$$

where $\mathscr{P}^v$ denotes the projection over $\mathbb{P}_d^v$.

Considering in each $\mathbb{P}_d$ a basis as (4.4), $\mathbf{B}_{ir}$ is the matrix of B_{ir}.

For the sake of simplicity let us put $(x_i, y_i) = (0, 0)$ (so $D_{j,i} \equiv 0$, for all $j \geq 1$), $t_i \geq t_r$ (in the general case the proof needs only some more tedious calculations).

Let us consider $(p, q) \in \mathbb{P}_d^{d+1-t_i} \times \mathbb{P}_d^{d+1-t_r}$, then

$$p(x, y) = \sum_{j=0}^{t_i - 1} p^{(d-j)}(x, y), \qquad q(x, y) = \sum_{j=0}^{t_r - 1} q^{(d-j)}(x, y),$$

where $p^{(j)}$, $q^{(j)}$ are homogeneous polynomials of degree j.

We shall study the Kernel of B_{ir}.

Denoting $D_{j,r}$ by D^j, $B_{ir}(p, q) = 0$ implies

$$\begin{aligned}
p^{(d)} &\equiv -q^{(d)} \\
p^{(d-1)} &\equiv -D^1 q^{(d)} - q^{(d-1)} \\
&\vdots \\
p^{(d-t_i+1)} &\equiv -\frac{1}{(t_i-1)!} D^{t_i-1} q^{(d)} - \cdots - \frac{1}{(t_i-t_r)!} D^{t_i-t_r} q^{(d-t_r+1)} \\
0 &\equiv \frac{1}{(t_i)!} D^{t_i} q^{(d)} + \cdots + \frac{1}{(t_i-t_r+1)!} D^{t_i-t_r+1} q^{(d-t_r+1)} \\
&\vdots \\
0 &\equiv \frac{1}{(t_i+t_r-1)!} D^{t_i+t_r-1} q^{(d)} + \cdots + \frac{1}{(t_i)!} D^{t_i} q^{(d-t_r+1)}.
\end{aligned} \tag{4.5}$$

Each leading principal submatrix of

$$\begin{bmatrix} \frac{1}{(t_i)!} & \cdots & \frac{1}{(t_i-t_r+1)!} \\ \vdots & & \vdots \\ \frac{1}{(t_i+t_r-1)!} & \cdots & \frac{1}{(t_i)!} \end{bmatrix},$$

is non-singular (Lemma 4.2), then, noting that $D^j D^i = D^{i+j}$, we are able to rewrite the last t_r relations of (4.5) as

$$\begin{aligned}
0 &\equiv a_{11} D^{t_i} q^{(d)} + \cdots + a_{1t_r} D^{t_i-t_r+1} q^{(d-t_r+1)} \\
0 &\equiv 0 + a_{22} D^{t_i} q^{(d-1)} + \cdots \\
0 &\equiv 0 + \cdots + a_{t_r t_r} D^{t_i} q^{(d-t_r+1)},
\end{aligned} \tag{4.6}$$

where $a_{ii} \neq 0$, $i = 1, \ldots, t_r$.

If $(x_r, y_r) \neq (0, 0)$ then the relation

$$D^{t_l} q^{(s)} \equiv 0, \qquad s \geqslant t_l,$$

determines $(s+1-t_l)$ coefficients of $q^{(s)}$; since $t_l + t_r \leqslant d+1$, relations (4.6) determine Q coefficients of $q^{(d)} \cdots q^{(d-t_r+1)}$, while the first t_l relations of (4.5) determine P coefficients of $p^{(d)} \cdots p^{(d-t_l+1)}$, where

$$Q = \sum_{j=0}^{t_r-1} (d+1-j-t_l), \qquad P = \sum_{j=0}^{t_l-1} (d+1-j).$$

Summarizing,

$$\operatorname{rank} \mathbf{B}_{lr} = \sum_{j=0}^{t_l-1} (d+1-j) + \sum_{j=0}^{t_r-1} (d+1-j-t_l) = \sum_{j=0}^{t_l+t_r-1} (d+1-j). \quad \blacksquare$$

LEMMA 4.2. *For each $p \in \mathbb{N}$, the matrix*

$$\mathbf{H}_p = (h_{ij}), \qquad h_{ij} = \frac{1}{(i+j+p-1)!}, \qquad i, j = 1, ..., n.$$

is non-singular.

Proof. Det $\mathbf{H}_p = [p! \cdots (p+n-1)!]^{-1} \det \hat{\mathbf{H}}_p$, where

$$\hat{\mathbf{H}}_p = \begin{bmatrix} \frac{1}{p+1} & \frac{1}{p+2} & \cdots & \frac{1}{p+n} \\ \frac{1}{(p+1)(p+2)} & & & \frac{1}{(p+n)(p+n+1)} \\ \vdots & & & \vdots \\ \frac{1}{(p+1)\cdots(p+n)} & & \cdots & \frac{1}{(p+n)\cdots(p+2n-1)} \end{bmatrix}.$$

Let us denote by a_r the rth row of $\hat{\mathbf{H}}_p$ and let us consider the following algorithm:

Algorithm 4.1.

0. *Given* $a_1 \cdots a_n$,
1. $i = n, ..., 2$
   2. $h = i-1, ..., 1$
      1. $a_i = a_h - (i-h) a_i$

By induction it is easy to prove that, after step $h=t$, the algorithm provides

$$a_i=\frac{1}{(p+1)(p+2)\cdots(p+t-1)(p+i)}, \ldots,$$
$$\frac{1}{(p+j)\cdots(p+t+j-2)(p+i+j-1)}, \ldots,$$
$$\frac{1}{(p+n)\cdots(p+n+t-2)(p+n+i-1)} \qquad i=n, \ldots, 2.$$

Then, after step $h=1$, $a_{ij}=1/(p+i+j-1)$, i.e., the algorithm, by linear combination of rows, changes $\hat{\mathbf{H}}_p$ into the Hilbert matrix, which is non-singular. ▮

5. The Dimensions

It is well known [14], that a lower bound for the dimension of $S_n^k(\Omega, \Delta)$ is

$$\dim S_n^k(\Omega, \Delta) \geqslant \alpha+\beta E-\gamma. \tag{5.1}$$

Theorem 2.1 gives an upper bound for the same quantity for general rectilinear partitions. This upper bound agrees with (5.1) if Δ is a generalized quasi-cross-cut partition and n is large enough, so it establishes the dimension.

For proving Theorem 2.1 it is useful to introduce some additional notation. Given a ray (cross-cut) with endpoints P_t, P_r we will refer to it as the ray (cross-cut) P_tP_r. Let us consider the lexicographical arrangement in $\mathbb{R}^2$ (i.e., $(x_i, y_i)<(x_j, y_j)$, iff $x_i<x_j$, or $x_i=x_j$ and $y_i<y_j$) and let the interior vertices be ordered. For each edge l_{is} emanating from an interior vertex let us put (Fig. 1)

$$\rho_{is}=\begin{cases} 0, & \text{if } P_i<P_s \text{ and } l_{is} \text{ overlies a ray } P_tP_r,\ P_t\in\mathring{\Omega},\ P_t\leqslant P_i, \\ 1, & \text{if } P_i>P_s \text{ and } l_{is} \text{ overlies a cross-cut or a ray } P_tP_r, \\ & \qquad P_t\in\mathring{\Omega},\ P_t\geqslant P_i, \\ 2, & \text{if } l_{is} \text{ does not overlie a cross-cut or a ray}, \\ 3, & \text{if } P_i>P_s \text{ and } l_{is} \text{ overlies a ray } P_tP_r,\ P_t\in\mathring{\Omega},\ P_t<P_i, \\ 4, & \text{if } P_i<P_s \text{ and } l_{is} \text{ overlies a cross-cut or a ray } P_tP_r, \\ & \qquad P_t\in\mathring{\Omega},\ P_t>P_i, \qquad i=1, \ldots, v,\ s=1, \ldots, V. \end{cases}$$

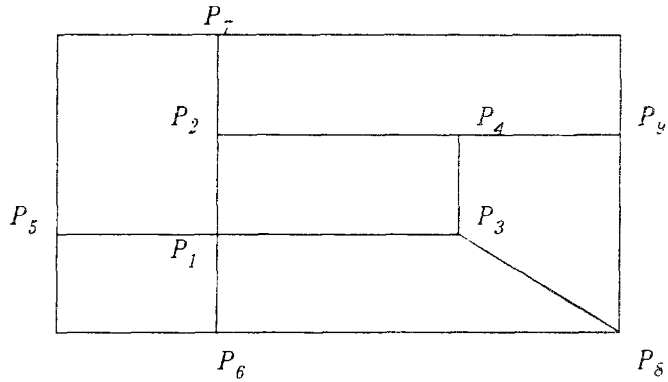

FIG. 1. $\rho_{24}=\rho_{38}=\rho_{49}=0$, $\rho_{15}=\rho_{16}=\rho_{21}=\rho_{31}=1$, $\rho_{34}=\rho_{43}=2$, $\rho_{42}=3$, $\rho_{12}=\rho_{13}=\rho_{27}=4$.

Proof of Theorem 2.1. From (4.1) it is sufficient to prove that

$$\operatorname{rank} \mathcal{M} \geqslant \beta E_{cd} + \gamma + \sum_{(i,s)\in I^d} n_{is}. \tag{5.2}$$

Let us construct a set $\mathscr{C}$ of columns of $\mathcal{M}$ according to the following steps $(i=1, ..., v)$:

(1) Select T $(T=\min(k+1+j, je_i))$ independent columns in $\mathbf{M}_j^i$ [15], $j=1, ..., n-k$, choosing at first all the possible columns associated to the edges l_{is} with $\rho_{is}=0$, after the ones associated to the edges with $\rho_{is}=1$ and so on until the amount T is reached.

(2) If $\rho_{is}=3, 4$ and P_s is an interior vertex, choose the β columns associated to l_{is} (j columns in each $\mathbf{M}_j^i$, $j=1, ..., n-k$).

(3) If $(i, s)\in I^d$ (hence P_i, $P_s \in \mathring{\Omega}$), choose the j columns in $\mathbf{M}_j^i$, $j=J_i, ..., n-k$, associated to l_{is} and a set of Q, $Q=j-(n-k+1-J_i)_+$, columns in $\mathbf{M}_j^s$, $j=\max(J_s, n-k+2-J_i), ..., n-k$ associated to l_{si}.

We call columns of type (i) those chosen at the step i, $i=1, 2, 3$, of the previous procedure.

Let $\hat{\mathcal{M}}$ be the submatrix of $\mathcal{M}$ consisting of the columns of $\mathscr{C}$ and let $\hat{\mathbf{M}}_j^i$, $\hat{\mathbf{L}}_i$, $\hat{\mathbf{L}}$ be respectively the submatrices of $\mathbf{M}_j^i$, $\mathbf{L}_i$, $\mathbf{L}$ consisting of the columns which are part of columns in $\mathscr{C}$.

Let us compute the cardinality of $\mathscr{C}$. We have

γ columns od type (1),

βE_{cd} columns of type (2),

$\sum_{(i,s)\in I^d} n_{is}$ columns of type (3), where

$$n_{is} = \sum_{j=J_i}^{n-k} j + \sum_{j=\max(J_s, n-k+2-J_i)}^{n-k} (j-(n-k+1-J_i)_+).$$

Since for every edge l_{is} such that $\rho_{is}=3$ (4) there exists one edge l_{ir}, emanating from P_i, collinear to l_{is}, with $\rho_{ir}=0$ (1), $i=1, ..., v$, in $\mathscr{C}$ there are no columns of type (1) associated to the edges such that $\rho_{is}=3$ (4). In addition we observe that, because of the ordering chosen in (1), in $\hat{\mathbf{M}}_j^i$, $j \geqslant J_i$, $i=1, ..., v$, there are no columns of type (1) associated to the edges not overlying cross-cuts or rays ($\rho_{is}=2$). Then the sets of columns (1), (2), (3) are disjoint and the cardinality of $\mathscr{C}$ agrees with the right hand side of (5.2).

We shall prove now that $\mathscr{C}$ consists of linearly independent columns of $\mathscr{M}$.

Let us assume that a linear combination of the elements in $\mathscr{C}$ is equal to zero: we shall prove that all the coefficients are zero.

Let us examine at first the columns associated to the edges such that $\rho_{is}=2$.

If $j<J_i$, in $\hat{\mathbf{M}}_j^i$, $i=1, ..., v$, there are only columns of type (1) and (2). Then, because of the ordering chosen in (1), the columns of type (1) associated to the edges not overlying cross-cuts or rays still remain independent on the other ones in $\hat{\mathbf{M}}_j^i$, $j<J_i$. It follows that their coefficients in the linear combination are zero, because of the structure of $\hat{\mathscr{M}}$.

Let us consider now the columns of type (3) in $\mathscr{C}$. Such columns are present if there exist edges l_{is} joining two interior vertices, not overlying a cross-cut or a ray and such that $\min(J_i, J_s) \leqslant n-k$. Let us examine in $\hat{\mathbf{L}}$ the columns related to any couple of these edges, l_{is} and l_{si}, i.e., the columns of $\hat{\mathbf{L}}_s$ and $\hat{\mathbf{L}}_i$. From the previous arguments it follows that among these columns the only ones having non-zero coefficients in the linear combination could be those which are part of columns of type (3) associated to l_{is} and l_{si}. Then we are dealing with the columns of matrix $\mathbf{B}_s$, $t_s=(n-k+1-J_i)_+$ (see Lemma 4.1) and with a subset of columns of matrix $\mathbf{B}_i$, $t_i=(n-k+1-\max(J_s, n-k+2-J_i))_+$. From Lemma 4.1 we can choose these columns in such a way that they are linearly independent because

$$(n-k+1-J_i)_+ + (n-k+1-\max(J_s, n-k+2-J_i))_+ \leqslant n-k.$$

This implies, recalling the structure of $\hat{\mathbf{L}}$, that all the columns of type (3) in $\mathscr{C}$ have zero coefficients in the linear combination.

Summarizing, all the columns in $\mathscr{C}$ associated to edges not overlying a cross-cut or a ray have zero coefficients in the linear combination.

Let us consider now the columns in $\mathscr{C}$ associated to the edges overlying cross-cuts or rays such that $\rho_{is} = 1, 4$. This we do by starting from the edges crossing P_v.

In $\hat{\mathbf{M}}_j^v$, $j = 1, ..., n-k$, there are no columns of type (2) associated to the edges with $\rho_{vs} = 4$ because there are no interior vertices grater than P_v. Then, because of the ordering chosen in (1), if there exist in $\hat{\mathbf{M}}_j^v$ columns of type (1) associated to edges with $\rho_{vs} = 1$ they are independent on the other columns which can have non-zero coefficients in $\hat{\mathbf{M}}_j^v$, $j = 1, ..., n-k$. It follows that their coefficients in the linear combination are zero, because of the structure of $\hat{\mathcal{M}}$.

Let us consider now the β columns in $\mathscr{C}$ associated to any edge l_{iv} with $\rho_{iv} = 4$, if it exists. Since we have a zero linear combination in $\hat{\mathcal{M}}$ we have a zero linear combination in $\hat{\mathbf{L}}$ too, particularly in $(\hat{\mathbf{L}}_v \mid \hat{\mathbf{L}}_i)$. Since $\rho_{vi} = 1$ each column associated to l_{vi} has zero coefficient; then, as every $\mathbf{L}_i$ is a non-singular matrix, each column associated to l_{iv} must have coefficient equal to zero in the linear combination, because of the structure of $\hat{\mathbf{L}}$.

Summarizing, all the columns in $\mathscr{C}$ associated to the edges crossing P_v such that $\rho_{vs} = 1, 4$ have zero coefficients in the linear combination.

Examining one after the other the vertices P_i, $i = v-1, ..., 1$, we can prove in the same way that all the columns in $\mathscr{C}$ associated to the edges such that $\rho_{rs} = 1, 4$ have zero coefficients in the linear combination.

Finally the only columns in $\mathscr{C}$ having non-zero coefficients in the linear combination could be the ones associated to the edges overlying rays such that $\rho_{is} = 0, 3$. In order to prove that these ones have zero coefficients as well, we can repeat the previous arguments considering at first the edges crossing P_1, and successively that ones crossing P_i, $i = 2, ..., v$.

Briefly every column in $\mathscr{C}$ has zero coefficient in the considered linear combination. ∎

As a consequence of the previous theorem we have:

Proof of Theorem 2.2. If Δ is a generalized quasi-cross-cut partition then

$$|I^d| = E_d - E_{cd},$$

while from (2.2), $n_{is} = \beta$, for each $(i, s) \in I^d$. Hence from Theorem 2.1, rank $\mathcal{M} \geqslant \beta E_d + \gamma$, then from (4.1),

$$\dim S_n^k(\Omega, \Delta) \leqslant \alpha + \beta E - \gamma. \tag{5.3}$$

The statement follows by comparing (5.3) and (5.1). ∎

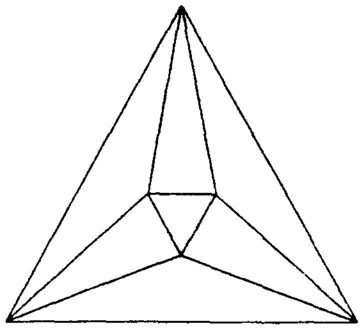

FIG 2. The Morgan Scott example. $v=3$, $\eta=\eta_i=2$, $i=1,2,3$, $E=9$, $E_d=3$, $\dim S_n^k(\Omega,\Delta)=\alpha+\beta E-\gamma\ \forall n\geqslant 3k$.

6. Remarks and Examples

Remark 6.1. If Δ is a quasi-cross-cut partition then $I^d=\emptyset$, $E_d=E_{cd}$ so (2.1) agrees with (5.1) and it establishes the dimension of the space for each n, k according to [5].

Remark 6.2. Theorem 2.2 holds in a little more general form. In fact from its proof it is easy to see that (2.1) agrees with (5.1) provided that

$$J_i+J_s+k-2\leqslant n, \qquad \text{or} \qquad \min(J_i,J_s)=1 \quad \forall(i,s)\in I^d.$$

These conditions can be substituted for (2.2).

Remark 6.3. For any generalized quasi-cross-cut partition Theorem 2.2 gives the dimension of $S_n^k(\Omega,\Delta)$ for each $n\geqslant 3k$, particularly for $S_3^1(\Omega,\Delta)$.

We end with some examples.

EXAMPLE 6.1. See Fig. 2.

EXAMPLE 6.2 [11]. See Fig. 3.

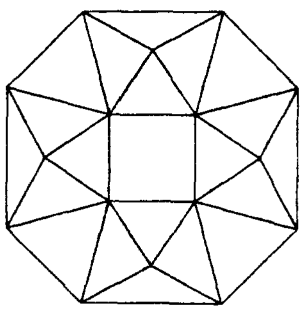

FIG. 3. $v=8$, $\eta=\eta_i=2$, $i=1,\ldots,8$, $\dim S_n^k(\Omega,\Delta)=\alpha+\beta E-\gamma\ \forall n\geqslant 3k$.

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